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# Test Sets for Integer Programs with $\mathbb{Z}$-Convex Objective

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**Abstract**

In this paper we extend test set based augmentation methods for integer linear programs to programs with more general convex objective functions. We show existence and computability of finite test sets for these wider problem classes by providing an explicit relationship to Graver bases. One candidate where this new approach may turn out fruitful is the Quadratic Assignment Problem.

## 1 Introduction

Integer linear optimization problems

$$(\mathrm{IP})_{c,b}: \qquad \min\{c^\intercal z : Az = b, z \in \mathbb{Z}_+^n\},$$

appear in many practical applications. One way to solve such a problem is to start with a feasible solution $z_0$ and to replace it by another feasible solution $z_0 - v$ with smaller objective value $c^\intercal(z_0 - v)$, as long as we find such a vector $v \in \mathbb{Z}^n$ that improves the current feasible solution. If the problem is solvable, that is in particular if it is bounded, this augmentation process has to stop (with an optimal solution).

The key step in this algorithmic scheme, besides finding an initial feasible solution, is to find improving vectors. Universal test sets, which depend only on the problem matrix $A$, provide such vectors for any given $c$ and $b$ and for any non-optimal feasible solution $z_0$ of $(\mathrm{IP})_{c,b}$. Note that universal test sets can in fact also be used to find an initial feasible solution $z_0$ [6]. For a survey on all currently known test sets for $(\mathrm{IP})_{c,b}$ see [12].

Graver [4] was the first to introduce a finite universal test set. The **Graver basis** $\mathcal{G}_{\mathrm{IP}}(A)$, or **Graver test set**, associated to $A$ consists of all $\sqsubseteq$-minimal *non-zero* solutions to $Az = 0$, where for $u, v \in \mathbb{Z}^n$ we say that $u \sqsubseteq v$ if $u^{(j)}v^{(j)} \geq 0$ and $|u^{(j)}| \leq |v^{(j)}|$ for all components $j = 1, \ldots, n$, that is, if $u$ belongs to the same orthant as $v$ and its components are not greater in absolute value than the corresponding components of $v$.

**Example 1.** Consider the problem

$$\min\{x + y : x, y \in \mathbb{Z}_+\}.$$

The Graver test set associated to the problem matrix $A = 0$ is $\{\pm(1, 0), \pm(0, 1)\}$. As one can easily check, already the subset $\{(1, 0), (0, 1)\}$ provides an improving direction to any non-optimal solution of this particular problem instance. Thus, with the help of $\{(1, 0), (0, 1)\}$, we can augment any given feasible solution to the (in this case unique) optimal solution $(0, 0)$. $\square$

Intrinsic to the proofs that there do exist finite (universal) test sets for $(\mathrm{IP})_{c,b}$ and that they do indeed provide an improving direction to any non-optimal feasible solution, is the fact that both the objective function and the constraints are linear. Now let us observe what happens with a non-linear objective function.

**Example 2.** Consider the problem

$$\min\{(x + y)^2 + 4(x - y)^2 : x, y \in \mathbb{Z}_+\}.$$

As again $A = 0$, the corresponding Graver basis is $\mathcal{G}_{\mathrm{IP}}(A) = \{\pm(1, 0), \pm(0, 1)\}$. However, this universal test set for the integer *linear* program $(\mathrm{IP})_{c,b}$ does not provide an improving direction to any non-optimal feasible solution for the *quadratic* problem given above:

Clearly, $(0, 0)$ is again the unique optimal solution with objective value 0. Now consider the point $(1, 1)$ with objective value 4. There are 4 points reachable from $(1, 1)$ via the directions given by $\mathcal{G}_{\mathrm{IP}}(A)$: $(1, 0)$ and $(0, 1)$, both with objective values 5, and $(2, 1)$ and $(1, 2)$, both with objective values 13. Therefore, in order to reach the optimum $(0, 0)$ from $(1, 1)$, additional vectors are needed in the test set.

As we will see below, the set $\mathcal{G}_{\mathrm{IP}}(A) \cup \{\pm(1, 1)\}$ provides improving directions to any non-optimal solution of the above quadratic problem. Moreover, this property remains true even if we change the objective function in a certain way. (For details see below.) For example, with the directions from $\mathcal{G}_{\mathrm{IP}}(A) \cup \{\pm(1, 1)\}$ we can also find the optimum of the following program:

$$\min\{e^{|x+y-3|} + 4(x - y + 2)^6 + 2x - y : x, y \in \mathbb{Z}_+\} \qquad \square$$

In this paper we relieve the restriction to linear objective functions and employ test set methods for the solution of integer optimization problems

$$(\mathrm{CIP})_{f,b}: \qquad \min\{f(z) : Az = b, z \in \mathbb{Z}^n_+\},$$

where $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, and where

$$f(z) := \sum_{i=1}^{s} f_i(c_i^\intercal z + c_{i,0}) + c^\intercal z.$$

Herein, $c \in \mathbb{R}^n$, $c_1, \ldots, c_s \in \mathbb{Z}^n$, $c_{1,0}, \ldots, c_{s,0} \in \mathbb{Z}$, and $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, s$, are $\mathbb{Z}$-convex functions with minimum at 0. We call $g : \mathbb{R} \to \mathbb{R}$ a **$\mathbb{Z}$-convex** function with minimum at $\alpha \in \mathbb{Z}$, if

the function $g(x+1)-g(x)$ is increasing on $x \in \mathbb{Z}$ and if $g(x+1)-g(x) \leq 0$ for all integers $x < \alpha$ and $g(x+1)-g(x) \geq 0$ for all integers $x \geq \alpha$. Clearly, these three conditions imply that $x=\alpha$ is a minimum of $g$ over $\mathbb{Z}$. We will, however, restrict our attention to $\mathbb{Z}$-convex functions with minimum at 0. This is no restriction, since we can transform any $\mathbb{Z}$-convex function $g$ with minimum at $\alpha$ to one with minimum at 0 by considering $\bar{g}(x) = g(x+\alpha)$ instead.

The problem type $(\text{CIP})_{f,b}$ includes for example linear integer programs for $f_1 = \cdots = f_s = 0$, or quadratic integer programs for $f_i(x) = x^2$. However, one could apply our approach also to more exotic functions as $f_i(x) = |x|$ or $f_i(x) = -x$ for $x \leq 0$ and $f_i(x) = e^x$ for $x > 0$, that is, the functions $f_i$ considered as functions from $\mathbb{R}$ to $\mathbb{R}$ need not be continuous.

Our main result is the following.

**Theorem 1.1** *Let $A \in \mathbb{Z}^{d \times n}$ and $c_1, \ldots, c_s \in \mathbb{Z}^n$ be given. Denote by $C$ the $s \times n$ matrix whose rows are formed by the vectors $c_1^\intercal, \ldots, c_s^\intercal$. Moreover, let $I_s$ denote the $s \times s$ unit matrix. Then for any particular choice*

- *of $\mathbb{Z}$-convex functions $f_1, \ldots, f_s$ with minima at $x = 0$,*
- *of $c_{1,0}, \ldots, c_{s,0} \in \mathbb{Z}$, and*
- *of $c \in \mathbb{R}^n$,*

*the set*

$$\mathcal{H}_{CIP}(A, C) := \phi_n \left( \mathcal{G}_{IP} \begin{pmatrix} A & 0 \\ C & I_s \end{pmatrix} \right)$$

*provides an improving direction to any non-optimal feasible solution of the problem $(CIP)_{f,b}$. Herein, $\phi_n$ defines the projection of a vector onto its first $n$ components, and for a set $G$ of vectors $\phi_n(G)$ denotes the set of images of elements in $G$ under $\phi_n$.*

Trivially, $\mathcal{G}_{\text{IP}}(A) \subseteq \mathcal{H}_{\text{CIP}}(A, C)$ for any matrix $C$. However, as we have seen in Example 2, this inclusion can be strict.

For $f_1 = \cdots = f_s = 0$, we simply obtain $\mathcal{H}_{\text{CIP}}(A, 0) = \mathcal{G}_{\text{IP}}(A)$ as a (universal) test set for $(\text{IP})_{c,b}$. But, as the following example shows, the set $\mathcal{G}_{\text{IP}}(A)$ gives improving directions even for a far bigger problem class.

**Example 3.** Consider the family of problems $(\text{CIP})_{f,b}$ where $c_1, \ldots, c_n$ are the unit vectors in $\mathbb{R}^n$, that is,

$$\min\{f(z) : Az = b, z \in \mathbb{Z}_+^n\}$$

with

$$f(z) := \sum_{i=1}^{s} f_i(z_i + c_{i,0}) + c^\intercal z.$$

As $C = I_n$, we need to compute the Graver basis of the Lawrence lifting

$$\begin{pmatrix} A & 0 \\ I_n & I_n \end{pmatrix}$$

of $A$. Since all elements in the kernel of this Lawrence lifting have the form $(u, -u)$ and since $(v, -v) \sqsubseteq (u, -u)$ in $\mathbb{Z}^{2n}$ if and only if $v \sqsubseteq u$ in $\mathbb{Z}^n$, this Graver basis is simply $\{(u, -u) : u \in \mathcal{G}_{\mathrm{IP}}(A)\}$. Thus, $\mathcal{H}_{\mathrm{CIP}}(A, I_n) = \mathcal{G}_{\mathrm{IP}}(A)$, showing that the set $\mathcal{G}_{\mathrm{IP}}(A)$ is also a test set for this bigger problem class where $A$ is kept fixed and the remaining problem data is allowed to vary. $\square$

Although test set based methods are not yet proven to be successful in practice, there is renewed hope from recent work on generating functions [1, 3], in which it is proved that in fixed dimension any given problem $(\mathrm{IP})_{c,b}$ can be solved via test sets in time polynomial in the input data. It would be an interesting research project to generalize this complexity result to certain classes of functions $f_i$, for example to $f_i(x) = \alpha_i x^{2\gamma_i}$ with $\alpha_i > 0$ and $\gamma_i \in \mathbb{Z}_+$.

The remainder of this paper is structured as follows: In Section 2 we show that our test set approach can be applied to convex quadratic optimization problems, of which the Quadratic Assignment Problem (QAP) is probably the most famous example. Finally, in Section 3 we prove our main theorem, Theorem 1.1.

## 2 Quadratic Programs

In this section we deal with the special case of convex quadratic optimization problems

$$\min\{z^\intercal Q z + c^\intercal z : Az = b, z \in \mathbb{Z}^n_+\},$$

where $Q$ is a symmetric, positive semi-definite matrix with only rational entries. These problems can be solved by the test set approach introduced in Section 1. The reason for this is the following basic result from the theory of quadratic forms [8].

**Lemma 2.1** *Let $Q \in \mathbb{Q}^{n\times n}$ be a symmetric matrix. Then there exist a diagonal matrix $D \in \mathbb{Q}^{n\times n}$ and an invertible matrix $U \in \mathbb{Q}^{n\times n}$ such that $Q = U^\intercal D U$. Moreover, each diagonal element $d_{ii}$ of $D$ is representable by the quadratic form $x^\intercal Q x$, that is, for all $d_{ii}$ there is some $x_i \in \mathbb{R}^n$ such that $d_{ii} = x_i^\intercal Q x_i$.*

**Corollary 2.2** *Let $Q \in \mathbb{Q}^{n\times n}$ be a symmetric positive semi-definite matrix. Then there exist a diagonal matrix $D \in \mathbb{Q}^{n\times n}$ with only non-negative entries and an invertible matrix $U \in \mathbb{Q}^{n\times n}$ such that $Q = U^\intercal D U$.*

**Proof.** This is an immediate consequence of Lemma 2.1, since $d_{ii} = x_i^\intercal Q x_i \geq 0$ for all $i$ as $Q$ is positive semi-definite. $\square$

Thus, every convex quadratic objective function $z^\intercal Qz$ can be restated as $\sum_{i=1}^{s}\alpha_i(c_i^\intercal z)^2$ with $\alpha_i > 0$ and $c_i \in \mathbb{Z}^n$. Therefore, the test set approach presented in Section 1 is applicable to these problems with $f_i(x) = \alpha_i x_i^2$, $\alpha_i > 0$. Moreover, we should point out that $s \leq n$, that is, the Graver basis that has to be computed for $\mathcal{H}_{\mathrm{CIP}}(A, C)$ involves at most $2n$ variables.

In the following, we will restrict our attention to quadratic 0-1 problems.

**Corollary 2.3** *Any quadratic* 0-1 *optimization problem*

$$\min\{z^\intercal Qz + c^\intercal z : Az = b, z \in \{0,1\}^n\}$$

*with symmetric matrix* $Q \in \mathbb{Q}^{n\times n}$ *can be rephrased as an equivalent problem*

$$\min\{z^\intercal \bar{Q}z + \bar{c}^\intercal z : Az = b, z \in \{0,1\}^n\},$$

*where* $\bar{Q} \in \mathbb{Q}^{n\times n}$ *is a symmetric, positive definite matrix.*

**Proof.** As $z_i^2 - z_i = 0$ for $z \in \{0,1\}$, the given optimization problem is equivalent to

$$\min\{z^\intercal Qz + c^\intercal z + \lambda(z^\intercal I_n z - \mathbf{1}^\intercal z) : Az = b, z \in \{0,1\}^n\},$$

where $\lambda \in \mathbb{R}$ denotes some fixed scalar. As for sufficiently large $\lambda = \bar{\lambda} \in \mathbb{Q}_+$ the matrix $Q+\bar{\lambda}I_n$ becomes positive definite, Lemma 2.2 can be applied, giving the result with $\bar{Q} = Q + \bar{\lambda}I_n$ and $\bar{c} = c - \bar{\lambda}\mathbf{1}$. $\square$

Consequently, *any* 0-1 quadratic optimization problem

$$\min\{z^\intercal Qz + c^\intercal z : Az = b, z \in \{0,1\}^n\}$$

can be written as

$$\min\{\sum_{i=1}^{s}\alpha_i(c_i^\intercal z)^2 + c^\intercal z : Az = b, z \in \{0,1\}^n\}$$

with $\alpha_i > 0$, and therefore the test set approach presented in Section 1 can be applied. However, choosing different $\bar{\lambda}$ in the proof of Corollary 2.3, we get different equivalent formulations for the same problem $(\mathrm{CIP})_{f,b}$. But as the following example shows, different problem formulations can lead to different test sets $\mathcal{H}_{\mathrm{CIP}}(A, C)$ for the *same* problem. These sets, however, are test sets for two *different problem families* of which the given specific problem is a *common member*.

**Example 4.** Consider the quadratic 0-1 problem with $A = 0$ and

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Since $A = 0$, we need to compute the Graver basis of $(C|I_3)$. But we have different choices for $C$. As $x_i^2 = x_i$, $i = 1, 2, 3$, we have

$$\begin{aligned} x^\intercal Q x &= 2x_1x_2 + 2x_1x_3 + 4x_2x_3 \\ &= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_1^2 - 2x_2^2 - 2x_3^2 \\ &= (x_1 + x_2 + x_3)^2 + (x_2 + x_3)^2 - x_1 - 2x_2 - 2x_3 \end{aligned}$$

and

$$\begin{aligned} x^\intercal Q x &= 2x_1x_2 + 2x_1x_3 + 4x_2x_3 \\ &= (x_1 - 2x_2 + x_3)^2 + (3x_1 + x_2 + 4x_3)^2 + 12(x_1 - x_3)^2 - 22x_1^2 - 5x_2^2 - 29x_3^2 \\ &= (x_1 - 2x_2 + x_3)^2 + (3x_1 + x_2 + 4x_3)^2 + 12(x_1 - x_3)^2 - 22x_1 - 5x_2 - 29x_3. \end{aligned}$$

Therefore, the corresponding two matrices for the test set computations are

$$\left( \begin{array}{c|c} C' & I_2 \end{array} \right) = \left( \begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right)$$

and

$$\left( \begin{array}{cc} C'' & I_3 \end{array} \right) = \left( \begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array} \right).$$

Using the software package 4ti2 [5], we obtain

$$\begin{aligned} \mathcal{H}_{\mathrm{CIP}}(A', C') &= \{(1,0,0), (0,1,0), (0,0,1), (1,-1,0), (0,1,-1), (1,0,-1)\} \\ \mathcal{H}_{\mathrm{CIP}}(A'', C'') &= \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,-1,0), (0,1,-1), \\ &\quad (0,1,1), (1,0,1), (1,0,-1), (1,1,-1), (1,1,1), (1,-1,1)\} \end{aligned}$$

Note that $\mathcal{H}_{\mathrm{CIP}}(A', C') \subsetneq \mathcal{H}_{\mathrm{CIP}}(A'', C'')$. □

This gives us much freedom to rewrite particular 0-1 problems, possibly arriving at much smaller test sets for the same problem. As the following example shows, the same phenomenon happens also in the general (non-0-1) case.

**Example 5.** Consider the problem with $A = 0$ and

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Again, since $A = 0$, we need to compute the Graver basis of $(C|I_s)$ for some integer $s$, and as the following shows, we have more than one choice for $C$:

$$\begin{aligned} x^\intercal Q x &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ &= (x_1 + x_2 + x_3)^2 + x_1^2 + x_2^2 + x_3^2 \\ &= (x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2 \end{aligned}$$

Corresponding to these two representations are the matrices

$$\left( \begin{array}{c|c} C' & I_4 \end{array} \right) = \left( \begin{array}{ccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

and

$$\left( \begin{array}{c|c} C'' & I_3 \end{array} \right) = \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right).$$

Using 4ti2 again, we obtain

$$\begin{aligned} \mathcal{H}_{\mathrm{CIP}}(A', C') &= \{(1,0,0),(0,1,0),(0,0,1),(1,-1,0),(0,1,-1),(1,0,-1)\} \\ \mathcal{H}_{\mathrm{CIP}}(A'', C'') &= \{(1,0,0),(0,1,0),(0,0,1),(1,-1,0),(1,0,-1),(0,1,-1), \\ &\quad (1,1,-1),(1,-1,1),(1,-1,-1)\} \end{aligned}$$

Note that again, $\mathcal{H}_{\mathrm{CIP}}(A', C') \subsetneq \mathcal{H}_{\mathrm{CIP}}(A'', C'')$. □

The quadratic assignment problem [2] deals with assigning $n$ facilities to $n$ locations such that a certain quadratic cost function is minimized. It can be formulated as the following problem involving permutation matrices :

$$\begin{aligned} \min\{\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n d_{ijkl} x_{ij} x_{kl} \quad + \quad & \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} : \\ & \sum_{j=1}^n x_{ij} = 1, \quad i \in \{1, \dots, n\}, \\ & \sum_{i=1}^n x_{ij} = 1, \quad j \in \{1, \dots, n\}, \\ & x_{ij} \in \{0, 1\}, \quad i, j \in \{1, \dots, n\}. \end{aligned}$$

The value $d_{ijkl}$ can be seen as costs for assigning facility $i$ to location $j$ **and** facility $k$ to location $l$, whereas $c_{ij}$ models a fixed cost incurred by locating facility $i$ to location $j$.

Even nowadays, QAP's of size $n > 30$ (that is, with more than only 900 binary variables) are still considered to be computationally extremely hard, if not intractable. One major problem in branch-and-bound algorithms that try to solve these problems is the lack of sharp lower bounds.

As we had seen after Corollary 2.2, our novel approach presented in Section 1 reduces the question of solving the QAP to finding a truncated Graver basis in at most $2n^2$ variables, of which $n^2$ variables are bounded by 1.

From a practical perspective, however, we can restrict our attention to certain orthants to find an improving vector to a given feasible 0-1 solution. Moreover, we can use the upper bound of 1. Besides speeding up the computation, both constraint types reduce drastically the number of test set vectors that could provide an improving direction to the current solution, a very important fact for practical applicability.

We think it to be an interesting future project to try our new test set approach to instances from the QAPLIB [10]. Although the software package 4ti2 [5] exploits both orthant and upper bound constraints, it does not yet include a special 0-1 implementation in which special data structures speed up the computation and save valuable memory.

# 3 Proof of Main Theorem

In this section we prove the main theorem, Theorem 1.1, of this paper. First, we will collect some facts about Graver bases that will turn out very useful in the final proof. Lemma 3.23 in [7] states the following.

**Lemma 3.1** *Let $B = \begin{pmatrix} A & a & -a \end{pmatrix}$ be an integer matrix such that the two columns $a$ and $-a$ differ only by a sign. Then the Graver basis of $B$ can be constructed from the Graver basis of $B' = \begin{pmatrix} A & a \end{pmatrix}$ in the following way:*

$$\mathcal{G}_{IP}(B) = \{(u, v, w) : vw \leq 0, (u, v - w) \in \mathcal{G}_{IP}(B')\} \cup \{\pm(0, 1, 1)\}.$$

A simple corollary of this is

**Corollary 3.2** *Let $B = \begin{pmatrix} A & a & a \end{pmatrix}$ be an integer matrix with two identical columns $a$. Then the Graver basis of $B$ can be constructed from the Graver basis of $B' = \begin{pmatrix} A & a \end{pmatrix}$ in the following way:*

$$\mathcal{G}_{IP}(B) = \{(u, v, w) : vw \geq 0, (u, v + w) \in \mathcal{G}_{IP}(B')\} \cup \{\pm(0, 1, -1)\}.$$

**Proof.** The claim follows immediately from the fact that $(u, v, w)$ is $\sqsubseteq$-minimal in $\ker \begin{pmatrix} A & a & a \end{pmatrix}$ if and only if $(u, v, -w)$ is $\sqsubseteq$-minimal in $\ker \begin{pmatrix} A & a & -a \end{pmatrix}$. □

The following is an immediate consequence of Lemma 3.1 and of Corollary 3.2.

**Lemma 3.3** *Let $A \in \mathbb{Z}^{d \times n}$ and let $B = \begin{pmatrix} A & a & \ldots & a & -a & \ldots & -a \end{pmatrix}$ be an integer matrix with finitely many multiple columns $a$ and $-a$ which differ only in their signs. Then we have $\phi_n(\mathcal{G}_{IP}(B)) = \phi_n\left(\mathcal{G}_{IP}\begin{pmatrix} A & a \end{pmatrix}\right) \cup \{0\}$.*

**Proof.** The constructions in Lemma 3.1 and in Corollary 3.2 satisfy $\phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{ccc} A & a & -a \end{array}\right)\right) = \phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{cc} A & a \end{array}\right)\right) \cup \{0\}$ and $\phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{ccc} A & a & a \end{array}\right)\right) = \phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{cc} A & a \end{array}\right)\right) \cup \{0\}$. Putting both constructions together iteratively, we get $\phi_n(\mathcal{G}_{\mathrm{IP}}(B)) = \phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{cc} A & a \end{array}\right)\right) \cup \{0\}$, as claimed. □

Thus, in order to compute $\phi_n(\mathcal{G}_{\mathrm{IP}}(B))$, it suffices to compute $\phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{cc} A & a \end{array}\right)\right)$. The following is an immediate consequence to Lemma 3.3.

**Corollary 3.4** *Let $A \in \mathbb{Z}^{d\times n}$, $c_1, \ldots, c_s \in \mathbb{Z}^n$, and $k \in \mathbb{Z}_{>0}$. Denote by $C$ the $s \times n$ matrix whose rows are formed by the vectors $c_1^\intercal, \ldots, c_s^\intercal$, by the bold letter $\mathbf{1}$ the vector in $R^k$ with all entries 1, and by $I_s$ the $s \times s$ unit matrix. Then*

$$\phi_n(\mathcal{G}_{IP}(A_k)) = \phi_n\left(\mathcal{G}_{IP}\left(\begin{array}{cc} A & 0 \\ C & I_s \end{array}\right)\right) \cup \{0\},$$

*where*

$$A_k := \left(\begin{array}{ccccccccc} A & & & & & & & & \\ c_1^\intercal & -\mathbf{1} & \mathbf{1} & & & & & & \\ c_2^\intercal & & & -\mathbf{1} & \mathbf{1} & & & & \\ & & & & & \ddots & \ddots & & \\ c_s^\intercal & & & & & & & -\mathbf{1} & \mathbf{1} \end{array}\right).$$

Before we come to the proof of our main theorem, let us prove two more useful facts.

**Lemma 3.5** *Let $g$ be a $\mathbb{Z}$-convex function with minimum at 0. Then for fixed $p \in \mathbb{Z}$ and for fixed $k \geq |p|$, an optimal solution to*

$$\min\{\sum_{j=1}^{k} \quad (g(j) - g(j-1))x_{i,j} + (g(-j) - g(-j+1))y_{i,j} :$$
$$p = \sum_{j=1}^{k} x_j - \sum_{j=1}^{k} y_j, \quad x_j, y_j \in \{0,1\}, \quad j = 1, \ldots, k\},$$

*is given by*

$$\begin{array}{ll} x_1 = \ldots = x_p = 1, x_{p+1} = \ldots = x_k = y_1 = \ldots = y_k = 0, & \textit{if} \quad p > 0, \\ x_1 = \ldots = x_k = y_1 = \ldots = y_k = 0, & \textit{if} \quad p = 0, \\ y_1 = \ldots = y_{-p} = 1, y_{-p+1} = \ldots = y_k = x_1 = \ldots = x_k = 0, & \textit{if} \quad p < 0. \end{array}$$

*The optimal value in each of these three cases is $g(p) - g(0)$.*

**Proof.** The case $p = 0$ is trivial and the optimal objective value is $0 = g(0) - g(0)$.

Let us now consider the case $p > 0$. Clearly, since $p > 0$, some $x_i$ must be positive. Suppose that in a minimal solution we have $x_i = 1$ and $y_j = 1$ for some $i$ and some $j$. This cannot happen, since by

putting $x_i = 0$ and $y_j = 0$ we would arrive at a solution with smaller objective value, as all coefficients in the objective function are positive. Thus, in a minimal solution $y_1 = \ldots = y_k = 0$.

Since $g$ is a $\mathbb{Z}$-convex function with minimum at 0, the coefficients $g(j) - g(j-1)$ in the objective function are non-negative and form an increasing sequence as $j > 0$ increases. Thus, $x_1 = \ldots = x_p = 1$, $x_{p+1} = \ldots = x_k = 0$ leads to a minimal objective value. This value is

$$\sum_{j=1}^{p}(g(j) - g(j-1)) = g(p) - g(0).$$

For the case $p < 0$ we conclude analogously that $x_1 = \ldots = x_k = 0$. Moreover, since $g$ is a $\mathbb{Z}$-convex function with minimum at 0, the coefficients $g(-j) - g(-j+1)$ in the objective function are non-negative and form an increasing sequence as $j > 0$ increases. As above, this implies that $y_1 = \ldots = y_{-p} = 1$, $y_{-p+1} = \ldots = y_k = 0$ leads to a minimal objective value. This value is again

$$\sum_{j=1}^{-p}(g(-j) - g(-j+1)) = g(p) - g(0)$$

and the claim is proved. $\square$

**Lemma 3.6** *Let $f_1, \ldots, f_s$ be $\mathbb{Z}$-convex functions with minimum at 0, $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, $c \in \mathbb{R}^n$, $c_1, \ldots, c_s \in \mathbb{Z}^n$, and $c_{1,0}, \ldots, c_{s,0} \in \mathbb{Z}$ be given. Then for fixed $z \in \mathbb{Z}^n$ and for fixed $k \geq \max\{|c_i^\intercal z + c_{i,0}|, i = 1, \ldots, s\}$, the optimal value of*

$$\begin{array}{lll}
\min\{\sum\limits_{i=1}^{s}\sum\limits_{j=1}^{k} & (f_i(j) - f_i(j-1))x_{i,j} + (f_i(-j) - f_i(-j+1))y_{i,j} + c^\intercal z : & \\
& Az = b, & z \in \mathbb{Z}_+^n, \\
& c_i^\intercal z + c_{i,0} = \sum\limits_{j=1}^{k} x_{i,j} - \sum\limits_{j=1}^{k} y_{i,j}, & i = 1, \ldots, s, \\
& x_{i,j}, y_{i,j} \in \{0,1\}, & i = 1, \ldots, s, \\
& & j = 1, \ldots, k\}.
\end{array}$$

*is $f(z) - \sum\limits_{i=1}^{s} f_i(0)$, where*

$$f(z) := \sum_{i=1}^{s} f_i(c_i^\intercal z + c_{i,0}) + c^\intercal z.$$

**Proof.** Since $z$ is fixed, the problem decomposes into $s$ smaller problems for which we can apply Lemma 3.5. Thus, the optimal value of the given problem is

$$\sum_{i=1}^{s}[f_i(c_i^\intercal z + c_{i,0}) - f_i(0)] + c^\intercal z = \sum_{i=1}^{s} f_i(c_i^\intercal z + c_{i,0}) + c^\intercal z - \sum_{i=1}^{s} f_i(0) = f(z) - \sum_{i=1}^{s} f_i(0).$$

$\square$

Now let us finally prove our main theorem, Theorem 1.1, introduced in Section 1.

**Proof.** In order to prove this claim, assume that we are given $\mathbb{Z}$-convex functions $f_1, \ldots, f_s$ with minimum at 0, $c_{1,0}, \ldots, c_{s,0} \in \mathbb{Z}$, $b \in \mathbb{Z}^d$, and $c \in \mathbb{R}^n$. Moreover, assume that we are given a non-optimal feasible solution $z_0$ to $Az = b$, $z \in \mathbb{Z}^n_+$.

The theorem is proved if we can find some vector $t \in \mathcal{H}_{\mathrm{CIP}}(A, C)$ such that $z_0 - t$ is feasible and such that $f(z_0 - t) < f(z_0)$. In the following, we construct such a vector $t$.

Since we assume $z_0$ to be non-minimal, there exists some better feasible solution $z_1$, say. Let

$$k := \max\{|c_i^\intercal z_0 + c_{i,0}|, |c_i^\intercal z_1 + c_{i,0}|, i = 1, \ldots, s\}$$

and consider the auxiliary integer linear program

$$\begin{array}{lll} (\mathrm{AIP}): & \min\{\sum\limits_{i=1}^{s}\sum\limits_{j=1}^{k} & (f_i(j) - f_i(j-1))x_{i,j} + (f_i(-j) - f_i(-j+1))y_{i,j} + c^\intercal z : \\ & & Az = b, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in \mathbb{Z}^n_+, \\ & & c_i^\intercal z + c_{i,0} = \sum\limits_{j=1}^{k} x_{i,j} - \sum\limits_{j=1}^{k} y_{i,j}, \qquad\qquad\qquad\quad i = 1, \ldots, s, \\ & & x_{i,j}, y_{i,j} \in \{0, 1\}, \qquad\qquad\qquad\qquad\qquad\qquad\quad i = 1, \ldots, s, \\ & & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad j = 1, \ldots, k\}. \end{array}$$

By Lemmas 3.5 and 3.6, the minimal values of (AIP) for fixed $z = z_0$ and $z = z_1$ are $f(z_0) - f_0$ and $f(z_1) - f_0$, where $f_0 = \sum\limits_{i=1}^{s} f_i(0)$. By $(z_0, x_0, y_0)$ and $(z_1, x_1, y_1)$ denote feasible solutions of (AIP) that achieve these values.

As $f(z_0) > f(z_1)$ by assumption, we have $f(z_0) - f_0 > f(z_1) - f_0$. Thus, $(z_0, x_0, y_0)$ is a feasible solution of (AIP) that is not optimal. Therefore, there must exist some vector $(t, u, v)$ in the Graver basis associated with the problem matrix of (AIP) that improves $(z_0, x_0, y_0)$. We will now show that $t \in \mathcal{H}_{\mathrm{CIP}}(A, C)$, that $z_0 - t$ is feasible for $(\mathrm{CIP})_{f,b}$, and that $f(z_0 - t) < f(z_0)$. The claim then follows immediately.

The problem matrix associated to (AIP) is

$$A_k := \begin{pmatrix} A & & & & & & & & \\ c_1^\intercal & \mathbf{-1} & \mathbf{1} & & & & & & \\ c_2^\intercal & & & \mathbf{-1} & \mathbf{1} & & & & \\ & & & & & \ddots & \ddots & & \\ c_s^\intercal & & & & & & & \mathbf{-1} & \mathbf{1} \end{pmatrix},$$

where

$$\phi_n(\mathcal{G}_{\mathrm{IP}}(A_k)) = \phi_n\left(\mathcal{G}_{\mathrm{IP}}\left(\begin{array}{cc} A & 0 \\ C & I_s \end{array}\right)\right) \cup \{0\} = \mathcal{H}_{\mathrm{CIP}}(A, C) \cup \{0\},$$

by Corollary 3.4. Therefore, $(t, u, v) \in \mathcal{G}_{\mathrm{IP}}(A_k)$ satisfies $t \in \mathcal{H}_{\mathrm{CIP}}(A, C) \cup \{0\}$. Moreover, as $(z_0, x_0, y_0) - (t, u, v)$ is feasible for (AIP), we must have $A(z_0 - t) = b$ and $z_0 - t \geq 0$, implying that $z_0 - t$ is feasible for $(\mathrm{CIP})_{f,b}$.

It remains to show $f(z_0 - t) < f(z_0)$, since this also implies $t \neq 0$ and hence $t \in \mathcal{H}_{\text{CIP}}(A, C)$.

Let $(z_0 - t, x_2, y_2)$ be a feasible solution of (AIP) that achieves the minimal value $f(z_0 - t) - f_0$ of (AIP) for fixed $z = z_0 - t$, see Lemmas 3.5 and 3.6 for its existence and construction. Clearly, this minimal objective value for fixed $z = z_0 - t$ is less than or equal to the objective value of $(z_0, x_0, y_0) - (t, u, v)$, which in turn is strictly less than $f(z_0) - f_0$, the objective value of $(z_0, x_0, y_0)$.

Therefore, $f(z_0 - t) - f_0 < f(z_0) - f_0$ and consequently $f(z_0 - t) - f(z_0)$. □

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